On Reduction of Critical Velocity in a Model of Superfluid Bose-gas with Boundary Interactions

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The existence of superfluidity in a 3D Bose-gas can depend on boundary interactions with channel walls. We study a simple model where the dilute moving Bose-gas interacts with the walls via hard-core repulsion. Special boundary excitations are introduced, and their excitation spectrum is calculated within a semiclassical approximation. It turns out that the state of the moving Bose-gas is unstable with respect to the creation of these boundary excitations in the system gas + walls, i.e. the critical velocity vanishes in the semiclassical (Bogoliubov) approximation. We discuss how a condensate wave function, the boundary excitation spectrum and, hence, the value of the critical velocity can change in more realistic models, in which "smooth" attractive interaction between the gas and walls is taken into account. Such a surface mode could exist in "soft matter" containers with flexible walls.

Keywords: Nonideal Bose-gas, Boundary Interactions, Surface Excitations, Critical Velocity, Superfluidity.

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I. INTRODUCTION

Recent experiments on Bose-Einstein Condensation (BEC) in magnetically trapped gases [1], [2], and excitons in semiconductor crystals and nanostructures [3], [4], [5] have made the subject of BEC more vital, more interdisciplinary. Many new questions appeared naturally as the understanding of the process of BEC progressed [6]. However, some "old" problems – such as the kinetics of BEC, the nature of superfluidity, the critical velocity problem, etc. – still remain the subject under consideration [7], (especially for new physical objects where the state of BEC was recently demonstrated [6]). Although the critical velocities are one of the first difficulties to be encountered in the study of superfluidity, they are still the least understood aspect in the theory [8], [9], [10].

This article is motivated mainly by the problem of critical velocity(-ies) in exciton superfluidity. It has been found experimentally [4], [11], [12] that a cloud of condensed excitons moves through a crystal with some constant velocity and some characteristic shape of the density profile. Several theoretical explanations of this anomalous transport have been put forward [13], [14], [15]. In spite of the fact that these explanations are based on different assumptions, there are several common ideas in the background of all these theories. For instance, it is the notion that interaction with a lattice is very important (if not to say crucial) in the BEC of excitons [15], [16], [17].

However, there are many outstanding questions that remain the subject of discussion [18]. One of such questions, for example, is the superfluid nature of exciton anomalous transport. In fact, it is not clear how the exciton condensate "feels" the boundary of the crystal (via interaction with surface phonons, e.g.,) or the impurities and other lattice imperfections that can bound an exciton. Generally speaking, clarification of the role of these friction sources may be essential for the understanding of the exciton superfluidity and the nature of critical velocities.

We approach the critical velocity problem by working out a simple model in which dilute 3D Bose-gas moves in a channel and interacts with the walls of this channel. The walls are modeled as two 3D solid bodies with well-defined boundaries. Although we take into account repulsive interaction between the particles of the gas, the proposed model cannot describe, for example, the superfluid He, which is a Bose liquid with strong interparticle interaction. Yet this is not the aim of this article. The main goal of this study is to explore the space of manæuvre appearing in the framework of the well known simple models, such as the weakly nonideal Bose-gas, if we switch on the gas-wall boundary interaction.

We show that the existence of the repulsive interactions between the Bose-gas and the channel walls leads to the essential reduction of the critical velocity of the superflow. The finiteness of the Landau critical velocity for the bulk (i.e. Bose-gas) excitations turns out not to be a sufficient condition of superfluidity. Note that in the present work we investigate the boundary excitations. Although the breakdown of the superfluidity is assumed to be accompanied

by vortex emission, we leave for future studies the questions of the vortex formation and their dynamics in the case when interaction with walls is taken into account.

II. CRITICAL VELOCITY PROBLEM

A closed system cannot undergo an inner macroscopic motion in thermodynamic equilibrium. Once such a motion is present, the system must evolve toward an equilibrium state. However, unless this transition is kinematically prohibited, (i.e. incompatible with conservation laws), the macroscopic motion is sustained. Such is the case with a small object moving without any viscous drag in stationary superfluid [19], [20]. This object is assumed to have no inner degrees of freedom, so that its momentum and energy depend only on the velocity $\mathbf{v}$ of the center of mass.

If the conservation laws for the creation of an excitation with the energy $\epsilon_{\text{gas}}(k)$ and the momentum $\mathbf{k}$ in a fluid (gas) lead to [19]- [21]

$$\epsilon_{\text{gas}}(k) > 0, \text{ in the object reference frame,} \tag{1}$$

the particle will continue to move without any experience of drag forces. Condition (1) is known as the Landau criterion of superfluidity. In fact, Eq. (1) holds that [19]- [21]

$$v < v_{\mathbf{L}} = \left(\epsilon_{\text{gas}}(k)/k\right)_{\min}. \tag{2}$$

Here $v_{\mathbf{L}}$ is the Landau critical velocity.

Formula (2) is in agreement with experiments performed with the semimicroscopic objects moving in the liquid helium [21].

Formula (1), (taken in the channel reference frame), is employed as a criterion of Bose-gas superfluid flow in channels. In that case, the channel walls are regarded as a massive macroscopic body in the above consideration (i.e. the walls act as some source of perturbations on the gas flow), and the final result is formulated in the form (2).

On experiments with liquid helium flow, however, the registered values of the critical velocities turn out to be much smaller than $v_{\mathbf{L}}$. Moreover, the critical velocity depends on the channel dimensions [22]. The fact that the liquid superfluid helium could not be treated as a dilute Bose-gas is believed to be the main reason for this discrepancy. It is generally assumed that the critical velocities are related to the appearance of quantized vortex lines in the superfluid. The Landau criterion (1) applied to the vortex excitations [23] can explain the critical effects in circular geometries. However, it cannot account for the drastically different critical velocities for rotation and linear flows [24].

A superflow of a dilute Bose gas, described by nonlinear Schrödinger equation [25], has been studied recently in different geometries with the use of direct numerical methods [26]- [27]. It has been observed that the distinct critical velocity is linked to the emission of vortices. This velocity turns out to be equal to the Bogoliubov [28] velocity of sound propagation in the gas. It is in good agreement with the criterion (1) since the Dirichlet boundary conditions (for Bose-gas wave function) are imposed on the channel walls [27]. This means that the walls, being considered as rigid immovable bodies, have no degrees of freedom. As a consequence, the arguments leading to formulas (1),(2) can be used.

In reality, the channel walls have a large number of degrees of freedom. Indeed, short- and long-range forces between a particle and a surface, boundary and interface phonons are well known subjects in the surface physics [29], [30]. This means that the (superfluid) Bose-gas is coupled with the channel, in which the gas moves. Then special boundary excitations can exist in the system of Bose-gas + channel walls because of the coupling between, say, the surface phonons of the walls and the Bogoliubov phonons of the Bose-gas. Therefore, the Landau criterion in the form (1) cannot be applied; it has to be modified. To get an analog of it we use the laws of conservation, taking the (inner) walls' degrees of freedom into consideration. The superflow can exist, provided the following condition holds:

$$\epsilon(k) > 0, \text{ in the channel (laboratory) reference frame,} \tag{3}$$

where $\epsilon(k)$ is the energy of *any* elementary excitation of the *whole* (gas + walls) system.

Condition (3) means that the state of the system can not be changed, since the occurrence of any number of elementary excitations leads to the increase of a total energy, but the latter is prohibited by the low of conservation. The excess of the momentum is "absorbed" by the motion of the center of mass of the walls. The energy is not actually changed by this motion (in the channel reference frame) because of a large mass of the walls and their zero initial velocity.

III. BOSE-GAS WITH BOUNDARY INTERACTIONS

We study the model of the dilute Bose-gas in the channel of the width $2l$ (in y-direction) and of infinite length (in x- and z-directions). The channel walls occupy $|y| > l$ part of the space (see Fig. 1). The general structure of the Hamiltonian is the following:

$$\hat{H} = H_{\mathrm{gas}}(\hat{\psi},\, \hat{\psi}^{\dagger}) + H_{\mathrm{ph}_1}(\hat{q}_1,\, \hat{\pi}_1) + H_{\mathrm{int}\,1}(\hat{q}_1,\, \hat{\psi}^{\dagger}\hat{\psi}) + \\ + H_{\mathrm{ph}_2}(\hat{q}_2,\, \hat{\pi}_2) + H_{\mathrm{int}\,2}(\hat{q}_2,\, \hat{\psi}^{\dagger}\hat{\psi}), \tag{4}$$

where $\hat{\psi}$ is the Bose-gas field operator, $\hat{q}$ is the displacement field operator of a wall, $\hat{\pi}$ is the momentum density operator conjugate to $\hat{q}$, and the indexes 1,2 correspond to the upper and lower part of the channel respectively. In the model being considered, the Bose-gas Hamiltonian has the following form:

$$H_{\mathrm{gas}} = \int \hat{\psi}^{\dagger}(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\Delta\right) \hat{\psi}(\mathbf{r})\, d\mathbf{r} + \int \frac{\nu}{2} \delta(\mathbf{r}-\mathbf{r}')\hat{\psi}^{\dagger}(\mathbf{r})\hat{\psi}^{\dagger}(\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r})\, d\mathbf{r}d\mathbf{r}',$$

where $\nu > 0$ is the interparticle interaction constant [20], and the wall Hamiltonian can be written as follows

$$H_{\mathrm{ph}} = \int \frac{\hat{\boldsymbol{\pi}}^2(\mathbf{r})}{2\rho} \;+\; \partial_j \hat{q}_k(\mathbf{r}) \lambda_{jkln} \partial_l \hat{q}_n(\mathbf{r})\, d\mathbf{r},$$

where the tensor λ_{jkln} describes the elastic properties of the channel walls.

We derive the excitation spectrum using the techniques of semiclassical approximation (cf. [28], [31], [32], [33]). Expanding the field operators near certain classical solutions, i.e. $\hat{\psi} = \psi_0 + \delta\hat{\psi}$ and $\hat{q} = q_0 + \delta\hat{q}$, we represent the Hamiltonian in the form

$$\hat{H} = H_0 + \hbar \hat{H}_2 + \ldots, \tag{5}$$

where H_0 stands for the classical part of $\hat{H}$. Note that $\psi_0 \neq 0$ indicates existence of a condensate in the moving Bose-gas, whereas $q_0 \neq 0$ appears in this model mainly to satisfy boundary conditions (see below). The Hamiltonian $\hat{H}_2$ in (5) is bilinear with respect to the field operators. As a consequence this (semiclassical) Hamiltonian can be reduced to the normal form

$$\hat{H}_2 = \sum_i \omega_i \hat{b}_i^{\dagger} \hat{b}_i + \mathrm{const},\; \left[\hat{b}_i, \hat{b}_j^{\dagger}\right] = \delta_{ij},\, \left[\hat{b}_i, \hat{b}_j\right] = 0. \tag{6}$$

The quantum Heisenberg and the classical Poisson-Hamilton equations of motion for the field operators (functions)

$$i(d\hat{b}_i/dt) = \left[\hat{H}, \hat{b}_i\right], \quad (db_i/dt) = \{H, b_i\}\,, \tag{7}$$

have the same form, if in (5) we neglect the terms of power in $\hbar$ greater than one. These equations are linear with respect to the field variables.

It follows from (6), (7) that the excitation energies ω_i are equal to the characteristic frequencies of these equations. Thus, to determine the semiclassical energy spectrum of the system we need to find the characteristic frequencies of the classical field equations for $\delta\psi$ and δq linearized around a proper stationary solution ψ_0, q_0. (This means that $\delta\psi$ and δq, originally of the operator nature, can be treated as a c-number and a real number function respectively). Then the gas state can be characterized by the Bose wave function $\psi(\mathbf{r}, t) = \psi(x, y, t)$, while the state of the walls is determined by the displacement field $\mathbf{q}(\mathbf{r}, t) = (q_x(x, y, t),\, q_y(x, y, t),\, 0)$. For simplicity, the system is asumed to be homogeneous in the z-direction.

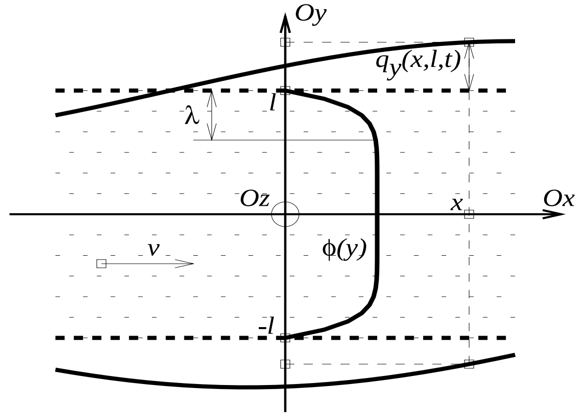

FIG. 1. The Bose gas moves with the velocity $\mathbf{v}$ in a channel of the width $2l$. The profile of the stationary wave function $\phi(y) = |\psi_0(x, y, t)|$ is shown in the center, where λ is the coherence length. The top and bottom curves depict the boundaries of the walls interacting with the gas, while the horizontal bold dashed lines correspond to the unperturbed boundaries, and $q_y(x, \pm l, t)$ are deviations from this equilibrium.

The interaction between the wall and gas atoms is given in our model by the sum of hard-core repulsion and some "smooth" potential $\mathcal{U} = \mathcal{U}(\mathbf{r},\, \psi^*\psi,\, \mathbf{q})$. The hard-core repulsion makes the walls impenetrable for the gas. It follows that the wave function of the Bose-gas vanishes along the actual boundaries $y_\pm = \pm l + q_y(x, \pm l, t)$ (see Fig. 1).

$$\psi\left(x, \pm l + q_y(x, \pm l, t), t\right) = 0. \tag{8}$$

The other boundary conditions, namely

$$\sigma_{yy}(x, \pm l, t) = \frac{\hbar^2}{m} \partial_y \psi^* \, \partial_y \psi(x, y = \pm l, t), \tag{9}$$

$$\sigma_{xy}(x, \pm l, t) = 0, \tag{10}$$

correspond to the equality of the forces between the solid and gas on the boundary. In (9), (10) σ_{ij} denotes the wall stress tensor and m is a mass of the gas atom. The pressure of the gas (r.h.s. of (9)) is equal to the normal component of the stress tensor (l.h.s. of (9)). Eq. (10) follows from the fact that the tangent stress vanishes on the boundary.

The wave function ψ of the repulsive Bose-gas satisfies the nonlinear Schrodinger equation [25]

$$\left\{ i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta - \nu\psi^*\psi \right\} \psi = V(\mathbf{r},\, \partial_j q_k)\, \psi, \;\; |y| < l, \tag{11}$$

while the wall dynamics obeys the hyperbolic equation for the displacement field $\mathbf{q}$ [34],

$$\rho \partial_t^2 q_i = \partial_j \sigma_{ij} - \partial_i W(\mathbf{r},\, \psi^*\psi), \quad |y| > l, \tag{12}$$

(ρ denotes the wall mass density). The potentials V and W depend on the "smooth" part of the gas-wall interaction $\mathcal{U}$; they must vanish if $\mathcal{U} = 0$. To make our model as simple as possible, we set $\mathcal{U} = 0$ *a priori*. Then, dynamics is described by equations (11), (12) with constant coefficients, whereas the hard-core interactions fix the boundary conditions (8)-(10). Note that these conditions imply that the walls have a finite compressibility, $K = -V(\partial_V\, p)_S < \infty$. Therefore, even though $\mathcal{U} = 0$, the wall and Bose-gas excitations can be coupled because $(\partial_y \psi)\, q_y(x, \pm l)$ in (8) and $\sigma_{yy} \sim K \partial_y q_y(x, \pm l)$ in (9) are finite and time dependent.

We suppose that the walls are isotropic (this assumption does not affect results qualitatively, simplifying our calculations), so that [34]

$$\sigma_{ij} = \rho c_t^2 \left(\partial_j q_i + \partial_i q_j + (\beta - 1)\delta_{ij}(\boldsymbol{\nabla}\mathbf{q}) \right), \; \beta = (c_l^2/c_t^2) - 1, \tag{13}$$

where c_l, c_t are longitudinal and transversal sound velocities respectively. It is convenient to rescale variables in such a way that the spatial coordinates are expressed via the coherence length λ units while the flow velocity is measured in terms of the Bogoliubov sound velocity c_{B} [28]:

$$\lambda = \hbar/\sqrt{\nu\rho_{\mathrm{gas}}}, \quad c_{\mathrm{B}} = \sqrt{\nu\rho_{\mathrm{gas}}}/m.$$

Here, $\rho_{\rm gas}$ stands for the bulk gas density. Then Eqs. (11),(12) become

$$(i\partial_t + (1/2)\Delta - \psi^*\psi)\,\psi(x,y,t) = 0, \tag{14}$$

$$\left\{c_t^{-2}\partial_t^2 - \Delta - \beta\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot)\right\}\mathbf{q}(x,y,t) = 0, \tag{15}$$

where the sound velocity, time and $\mathbf{q}$ are measured in the units of $c_{\rm B}$, $\lambda/c_{\rm B}$ and λ respectively.

In the stationary regime, the Bose gas moves uniformly in the x-direction with the velocity v. As the system is homogeneous in the x and z directions, $|\psi_0(\mathbf{r},t)| = \phi(y)$ and $\mathbf{q}_0(\mathbf{r},t) = \mathbf{q}_0(y)$. The walls are deformed only in the y-direction, since the tangent stress vanishes in the stationary regime $\mathbf{q}_0(y) = \big(0,\, q_{0y}(y),\, 0\big)$. The corresponding solution of (14), (15) is given by

$$\psi_0(x,y,t) = \phi(y)\mathrm{e}^{ivx}\,\mathrm{e}^{-i\Omega t},\ \ \phi(\pm l) = 0,\ \ \phi'(\pm l) \neq 0, \tag{16}$$
$$q_{0x} = 0,\ \ q_{0y} \equiv Q(y) = \pm\mathrm{const}(y \pm l),\ (y \to \pm l),\ \ Q(\pm l) = 0,\ \ Q'(\pm l) \neq 0 \tag{17}$$

It follows from (16), (14) that $\Omega = \tilde\mu + \frac{v^2}{2}$, $\tilde\mu = 1$ (this corresponds to the value of a chemical potential $\mu = \nu\rho_{\rm gas}/m$ at $T = 0$), and $\phi(y)$ satisfies the following equation [35]:

$$-\frac{1}{2}\phi'' + \phi^3 = \phi, \quad \phi(y) = \phi(-y). \tag{18}$$

The parity of ϕ in (18) and the boundary conditions in (16) are obtained from (8)-(10) (see Fig. 1).

We follow the procedure (5)-(7) expanding the field variables around the stationary solution (16), (17)

$$\psi = \Big(\phi(y) + \xi(x - vt, y, t)\Big)\mathrm{e}^{ivx}\,\mathrm{e}^{-i(\tilde\mu + v^2/2)t},$$
$$q_x = \zeta_x(x - vt, y, t),\ q_y = Q(y) + \zeta_y(x - vt, y, t).$$

Substituting these expansions into (14),(15) we get the following linear differential equations for fluctuations $\xi, \boldsymbol{\zeta}$,

$$\left\{i\partial_t + \frac{1}{2}\Delta + 1 - 2\phi(y)^2\right\}\xi - \phi(y)^2\xi^* = 0,\ |y| < l, \tag{19}$$
$$\left\{c_t^{-2}(\partial_t - v\partial_x)^2 - \Delta - \beta\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot)\right\}\boldsymbol{\zeta} = 0, \quad |y| > l, \tag{20}$$
$$\xi = \xi(x,y,t), \quad \boldsymbol{\zeta} = \boldsymbol{\zeta}(x,y,t),$$

written in the reference frame moving with the Bose-gas, $x \to x' = x - vt$.

One of the advantages of setting $\mathcal{U} = 0$ is the possibility of using the exact solution of Eq. (18) with the boundary conditions (16),

$$\phi(y) = \mathrm{sn}(y + l,\ \varrho),$$

where $\mathrm{sn}(y,\varrho)$ is the elliptic sine [36], the parameter ϱ is chosen to fit the boundary conditions and the following condition holds $\mathrm{sn}(y,\varrho) \to \tanh(y)$ if $l \to \infty$. (Notice that the dimensional condensate wave function can be written in the form $\phi_d(y) = \mathrm{const}\sqrt{\rho_{\rm gas}/m}\,\mathrm{sn}\big((y + l)/\lambda,\ \varrho\big)$).

Nontrivial excitations can not propagate over the wall region far from the boundary. Indeed, the equation (20) has the constant coefficients and, hence, the dispersion law for such excitations coincides with the phonon one (i.e. corresponding asymptotical solutions describe propagation of the ordinary sound waves far from the boundaries). Therefore, we have to look for the solution of (20) decreasing in $y \to \pm\infty$ directions. We assume that the experimentally discovered dependence of the critical velocity on the canal width [22], $v_c \sim l^{-n}$, $n \simeq 2$, is a hint to search for the special type of excitations, in which two boundaries of the canal can contribute *coherently.*

Such a solution of (19), (20) can be written in the form

$$\xi = \chi_1(y)\sin(kx - \omega t) + i\chi_2(y)\cos(kx - \omega t),\ \ |y| < l, \tag{21}$$
$$\zeta_x = r_1(y)\cos(kx - \omega t),\ \ \zeta_y = r_2(y)\sin(kx - \omega t),\ \ |y| > l \tag{22}$$

with

$$r_i(y) = A_i\exp(-\kappa|y|) + B_i\exp(-\eta|y|). \tag{23}$$

Two exponential terms in (23) correspond to the different polarizations of the boundary excitations. The characteristic values $\kappa = \sqrt{k^2 - \omega^2/c_t^2}, \eta = \sqrt{k^2 - \omega^2/c_l^2}$ are eigenvalues of the ordinary linear equations obtained by substitution of (22) into (20). Note that the ansatz (21), (22) is equivalent to the Bogoliubov u-v transformation generalized to a nonuniform case [28], [32], [33] and coupling with the surface phonons:

$$e^{i\mu t}\delta\psi(x,y,t) = u_k(y)\,\mathrm{e}^{i(kx-\omega(k)t)} + v_k^*(y)\,\mathrm{e}^{i(-kx+\omega(k)t)}$$
$$\delta\mathbf{q}(x,y,t) = \mathbf{C}_k(y)\,\mathrm{e}^{i(kx-\omega(k)t)} + \text{c.c.}.$$

Then the operators $b_k^\dagger$, (b_k) that create (annihilate) the boundary excitations (21-23) in the diagonalized Hamiltonian (6) can be represented by the linear combinations of the Bose-gas field operators, $\delta\hat{\psi}$ and $\delta\hat{\psi}^\dagger$, and the displacement field operators, $\delta\hat{q}$ and $\delta\hat{\pi}$.

We linearize the boundary conditions (8)-(10) according to the method of the semiclassical approximation (5), neglecting the terms of power greater than one in $\xi, \boldsymbol{\zeta}$. Together with the proper solution (21),(22) of Eqs. (19), (20), conditions (8)-(10) determine values $A = A(k,\omega)$, $B = B(k,\omega)$ in (23) and the boundary conditions for $\chi_{1,2}$:

$$\left(\frac{\chi_1'}{\chi_1}\right)_{y=\pm l} = \mp k\,\gamma(z),\;\; \chi_2(\pm l) = 0, \quad z = \frac{\epsilon(k)^2}{(c_t k)^2}, \tag{24}$$
$$\gamma(z) = \frac{\rho c_t^2/\rho_{\rm gas} c_{\rm B}^2}{\left(\phi'(l)\right)^2}\frac{1}{2z}\left(4\sqrt{1-z} - \frac{(2-z)^2}{\sqrt{1 - c_t^2 z/c_l^2}}\right) \gg 1.$$

The value $\epsilon(k)$ in (24)

$$\epsilon(k) = \omega(k) - kv$$

equals the boundary excitation energy in the channel reference system (3).

It follows from (21) and (19) that the variables $\chi_{1,2}(y)$ and ω satisfy the following system:

$$L_1\chi_1(y) + 2\omega\chi_2(y) = 0, \quad L_1 = \partial_y^2 - k^2 + 2 - 6\phi(y)^2,$$
$$L_2\chi_2(y) + 2\omega\chi_1(y) = 0, \quad L_2 = \partial_y^2 - k^2 + 2 - 2\phi(y)^2. \tag{25}$$

Note that in view of the parity of $\phi(y)$ (see (18)) and the boundary conditions (24), solutions of (25) can be either symmetric or antisymmetric with respect to y.

According to the criterion (3), a breakdown of superfluidity occurs at such a value v if there exists such a $\tilde{k} \neq 0$ that $\epsilon(\tilde{k}) = 0$. Then the argument z of γ in (24) vanishes and the boundary conditions do not depend explicitly on ω, i.e. $(\chi_1'/\chi_1)|_{y=\pm l} = \mp\tilde{k}\gamma(0) = \mp\text{const}$. In principle, this fact makes it possible to calculate v_c without finding any final expression of $\omega(k)$. Indeed, one has to solve Eqs. (25) with $\omega = \tilde{k}v_c$ and fixed boundary conditions.

In this simple model, however, it is possible to calculate the dispersion relation $\omega(k)$, at least in the $k \to 0$ limit [37]. We look for the symmetric solution of (25), expanding $\chi_{1,2}(y)$ in powers of ω^2:

$$\chi_1 = \chi_1^{(0)} + \omega^2\chi_1^{(1)} + \ldots, \quad \chi_2 = \omega(\chi_2^{(0)} + \omega^2\chi_2^{(1)} + \ldots).$$

The functions $\chi_{1,2}^{(i)}$ symmetric in y satisfy the following recurrence relations

$$L_1\chi_1^{(0)} = 0,\; \chi_1^{(0)}(\pm l) = 1,$$
$$L_1\chi_1^{(i)} + \chi_2^{(i-1)} = 0,\; \chi_1^{(i)}(\pm l) = 0,\; i = 1, 2, \ldots$$
$$L_2\chi_2^{(i)} + \chi_1^{(i)} = 0,\; \chi_2^{(i)}(\pm l) = 0,\; i = 0, 1, 2, \ldots. \tag{26}$$

The analytic study of (26) seems to be difficult. Instead we proceed numerically [38]; we obtain the eigenvalues $\omega = \omega(k)$ by solving (26) recursively and imposing the boundary conditions (24) on χ.

The result reads

$$\omega(k) = \alpha\sqrt{(\gamma + \delta)k^3}, \quad \text{as} \quad k \to 0, \tag{27}$$

where $\alpha > 0$ and $\delta > 0$ are some bounded functions of l, $\gamma = \gamma(0) \simeq \rho c_t^2/\rho_{\rm gas} c_{\rm B}^2$. The dependence of $\log(\omega(k))$ on $\log(k)$ is shown on Fig. 2. Note that only the solutions with finite $\gamma > 1$ have the physical meaning. For "conventional"

systems, such as a dilute Bose gas inside a solid container, the value of $\gamma(0)$ is very large, $\gamma(0) \simeq 10^{7\sim 8}$ or even more. Moreover, the validity of the long-wavelength / low-energy approximation implies $k\gamma < 1$ and $\omega \ll 1$, and the relevant wavelengths are unphysically huge. However, for the "soft matter" substances with flexible walls, γ can be of the order of $10^{2\sim 3}$ and the distance between the walls can be $2l > \lambda$. Then, beginning from the wavelengths of the order of $(10^{2.5\sim 3})\lambda$, we are within the '$k \to 0$' limit and $\omega(k) < 10^{-3}\tilde{\mu}$. (On Fig. 2, we present also the curves with $\gamma < 1$ because of similarity between our result and the dispersion relation of capillary waves on the interface between liquid and gaseous He, the so-called "ripplons" [39].) If the channel walls were rigid and incompressible, that corresponds to $\gamma = \infty$ in (27), the inhomogeneous surface excitations introduced in this study just do not exist as a well-defined object.

It is easy to conclude that the semiclassical critical velocity is *zero* in this *model*, since for any v there exist $k \neq 0$ such that $\epsilon(k) = \omega(k) - kv < 0$. This means that the model, in which all the gas − boundary interactions are reduced to the hard-core repulsion, predicts (in the semiclassical (Bogoliubov) approximation) an instability of the Bose-gas current state in relation to occurrence of boundary excitations. However, whether the damping of superflow can happen via the energy transfer from the 3D condensate to the boundary localized modes is an open question, which cannot be answered in the framework of models with the superfluid density $\rho_s = \rho_{\rm gas}$, $T = 0$. More sophisticated models of superfluidity, such as the two-fluid hydrodynamics [22], should be used to describe the kinetics of damping.

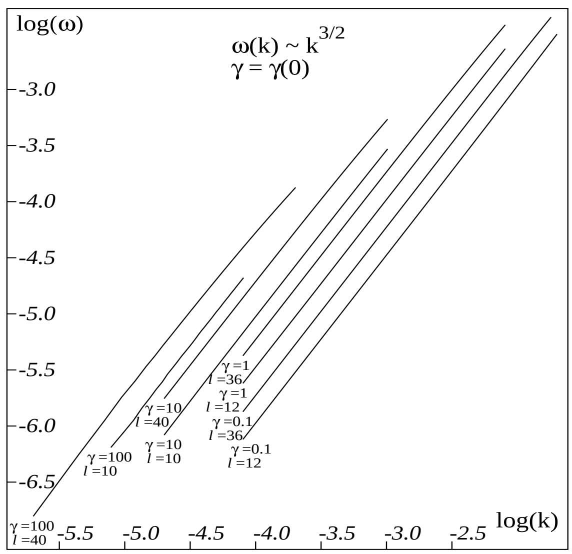

FIG. 2. The low-momentum spectrum $\omega(k)$ of boundary excitations. The canal width l is measured in the coherence length units; $\gamma \simeq \rho c_t^2/\rho_{\rm gas} c_{\rm B}^2$.

IV. DISCUSSION

In our opinion, the benefit from finding such a dispersion relation (see Fig. 2) in the framework of the proposed oversimplified model is the possibility of concluding that the boundary interactions can play a key role in the superfluidity, namely, by reducing substantially the critical velocity of the superflow. If that is the case, it is reasonable to generalize our model to make it more realistic.

Recall that we have neglected the "smooth" part $\mathcal{U}$ of the gas-wall interactions. The models with repulsive $\mathcal{U}$ seems to be qualitatively similar to the one under consideration. We believe that they also lead to the zero *semiclassical* critical velocity, since nothing can apparently prevent the energy transfer from the gas flow to the walls (in this approximation). It is important to note that even if the higher quantum corrections yield nonzero critical velocity, the latter will be of much smaller value than the (semiclassical) Bogoliubov-Landau velocity found by the numerical simulations based on the nonlinear Schrödinger equation [27].

The situation might be different if the "smooth" part $\mathcal{U}$ of the interaction between the gas and wall atoms were attractive. For example, we can take the Hamiltonian of gas-wall interaction in the Deformation Potential form

$$\hat{H}_{\rm int} = \int \sigma(\mathbf{r}, \mathbf{r}')\, \boldsymbol{\nabla}\hat{\mathbf{q}}(\mathbf{r})\, \hat{\psi}^\dagger \hat{\psi}(\mathbf{r}')\, d\mathbf{r}d\mathbf{r}', \tag{28}$$

where $\mathbf{r}$ and $\mathbf{r}'$ change in the wall area ($|y| > l$) and in the channel area ($|y'| < l$) respectively, and the function $\sigma(\mathbf{r}, \mathbf{r}')$ describes atom-lattice interaction. (At least two new parameters, which control the "smooth" part of gas-wall interaction, have to appear in the model with $\mathcal{U} \neq 0$: one, some characteristic value of energy, and, two, a length scale). Then the equations for the classical parts of $\hat{\psi}$ and $\hat{q}$, namely $\phi(y)$ and q_{0_y}, (see Eqs. (16), (17)), can be written as follows:

$$\left(\mu + \frac{\hbar^2}{2m}\partial_y^2 - \nu\phi^2(y)\right)\phi(y) = \left(\int \sigma(\mathbf{r}', \mathbf{r})\partial_{y'}Q(y')\, d\mathbf{r}'\right)\phi(y), \tag{29}$$

$$-\, c_l^2\, \partial_y^2 Q(y) = \rho^{-1}\partial_y \int \sigma(\mathbf{r}, \mathbf{r}')\, \phi^2(y')\, d\mathbf{r}'. \tag{30}$$

After the exclusion of $Q(y)$ from Eq. (29), it can be rewritten in the following form:

$$\left(-\frac{\hbar^2}{2m}\partial_y^2 + \Lambda\, U(\mathbf{r}) + \nu\phi^2(y) - \int U_{\text{eff}}(\mathbf{r}, \mathbf{r}')\phi^2(y')d\mathbf{r}'\right)\phi(y) = \mu\phi(y) \tag{31}$$

where $U_{\text{eff}}(\mathbf{r}, \mathbf{r}') = \int\sigma(\mathbf{r}'', \mathbf{r})\sigma(\mathbf{r}'', \mathbf{r}')d\mathbf{r}''/\rho c_l^2$, $U(\mathbf{r}) = \int\sigma(\mathbf{r}', \mathbf{r})d\mathbf{r}'$, and $\Lambda = \text{const}$ is defined from the boundary condition (9).

It is easy to see from the structure of (31) that the atom-lattice interaction $\sigma(\mathbf{r}, \mathbf{r}')$ (when exceeding a certain magnitude) can induce an attraction between the gas atoms in the boundary region. Indeed, there can exist such a scale of $|\mathbf{r} - \mathbf{r}'|$ that $\nu\delta(\mathbf{r} - \mathbf{r}') - U_{\text{eff}}(\mathbf{r}, \mathbf{r}') < 0$. In that case, one would expect essential changes in the spectrum of boundary excitations. The study of exciton superfluidity [15], [40] hints such a possibility.

This study reveals a mechanism of the exciton-exciton attraction induced by the lattice effects. The exciton gas falls into the soliton-like state ψ_0, q_0, when the exciton-phonon coupling constant exceeds a certain value, or, equivalently, the velocity v exceeds some critical value. The excitation spectrum in an exciton branch has a gap, and the Landau critical velocity, calculated for this type of excitations, is given by

$$v_c \simeq \hbar/mL, \tag{32}$$

where L denotes the characteristic size of the soliton. We can adapt the similar considerations in our model, replacing the exciton-phonon by the gas-wall interactions. The critical velocity can be given by a formula similar to (32) with $L \simeq l$, provided l is the only macroscopical length in the theory.

In this article we did not consider the influence of the long-range van der Waals forces between the walls and gas atoms on the stability of superfluid flow. Although the attractive part of these forces originates from interaction between the electron shells of the particles [41], it can be included in our model in the form of a static van der Waals potential appearing in the r.h.s. of Eq. (11). Such an external potential being localized near the boundaries can be stronger than the effective potential $\Lambda U(y)$ in (31) and therefore can change the properties of the condensate wave function, the boundary excitations, etc..

V. CONCLUSIONS

In conclusion, our simple model manifests one of the possible microscopic mechanisms for dissipation processes in the quantum Hamiltonian system of coupled Bose-gas and channel walls. We show that the dissipation can also be caused by creation of boundary excitations in this system. Although, in the semiclassical approximation, this process is not prohibited at any velocity of the moving repulsive Bose-gas, the higher quantum corrections to the self-energy part of the boundary excitations may be essential to obtain the nonzero value of v_c in the theory of the dilute Bose-gas with gas-wall interaction. On the other hand, more rigorous consideration in the framework of the semiclassical approximation should involve solutions of (8)-(12) with $\mathcal{U} \neq 0$, where the attractive part of gas-wall interaction is taken into account. Such solutions can be still represented in the form (21), (22), though equations (18), (19), (20) would become integral. A direct numerical study of the flow dynamics would be also useful.

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